CHAPTER 1

CHALLENGING PERSPECTIVES ON MATHEMATICS CLASSROOM COMMUNICATION

From Representations to Contexts, Interactions, and Politics

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We have chosen to put together this book for two reasons. First, we acknowledge dial an increased focus on communication has influenced and shaped work in a range of perspectives and areas in mathematics education research and practice. Over the past decade and a half, we have witnessed more and more studies that—although not directly concerned with analyses of communication—are considering communication as an integral part of pedagogy and didactics in mathematics classrooms, mathematics education curricula, and broader educational structures. The notion of communication opens up to embrace not only what is happening and what is being said by the participants in a classroom setting, but also conveys the

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underlying values, ideologies and politics that influence the practice, and thereby the formation of identities, the ways in which participants make sense of their experiences, what counts as valid activity, and the extent to which the participants can claim to belong to specific communities of practice. Second, we would like to initiate a dialogue concerning these areas of study, and to reflect on how they describe, define and conceive of communication. Despite recent developments, studies of mathematics classroom communication still largely assume particular perspectives on learning, on persons, on mathematics, and on research. Our main focus, as the title of i he book implies, is to challenge predominant perspectives and to highlight through the chapters, and through reflection and discussion over the chapters, what continue to be unanswered questions, the silenced issues, and the research challenges.

The chapters of this book are organized in three parts, Part I, containing this chapter, provides an introduction to the book, as well as a review of multiple perspectives of research about communication in mathematics classroom's. In this chapter, we place the chapters of the book in the context of aq overview of major studies concerning mathematics classroom communication over the past decades. This serves as a backdrop for a thematization of research in the area. Thus, we have formulated three i hemes. The first theme approaches communication mainly as register, representations, and contexts. The second theme places more emphasis on communication as social interactions, social setting, and classroom activity. Finally, the third theme introduces a new perspective of communication as practice, community, identity, and politics. These three themes provide a typology of major research perspectives relevant to mathematics classroom communication. They can also be seen as a tool to discuss how communication in mathematics classrooms has been researched over the last decades, and to highlight new directions. Having said that, we must emphasize that we do not approach these thematic areas as discrete; they interconnect. Part II contains nine contributing chapters, which are further organized into groups that represent the three themes outlined above. Part III contains three reflective commentaries. These commentaries are written by experienced academics in the field of mathematics education, Anna Sfard, Helle Aire, Ole Skovsmose, and Stephen Lerman, who all have an interest in language and communication as contexts to research and to conceive mathematical learning. Each of these commentaries provides a useful reflection across individual chapters and to the work of the book as a whole.

The three themes in Part II arc not viewed as discrete although they are presented separately. The first theme (Theme I: Register, Representations, and Contexts), deals with how representational media and language come into use when humans try to communicate their mathematical ideas. The issues discussed include: how knowledge production and communication are transformed by how media are used by humans and not media alone (Chapter 2); the influence of the integration of different representational media in geometry classrooms (Chapter 3); and the language use in developing and interpreting mathematical texts in different practices or situational contexts (Chapter 4).

Throughout this introduction, the word "context" is employed with a number of meanings, from referring to the recontextualization of everyday situations in themes or tasks, over referring to the context of particular discourses and practices of die mathematics classroom, to referring to the broader social, political and cultural context in which the school situation is embedded. We will preface our use of "context" throughout the text, using "task context," "situational context," and "political context" or "cultural context," respectively. To some extent, "situational context" and "socio-political/cultural context," will overlap with each other, as well as with the notions of "social setting" and "practice." Since the terms are used so extensively, we have chosen to include all of them, refraining from a lengthy discussion of possible differences in interpretation.

The second theme (Theme II: Social Interactions, Social Setting and Classroom Activity), focuses on how interactions between teacher and pupils or amongst pupils influence die patterns and values of communication in mathematical activity. The focus is on micro-analyses of classroom activity and the topics discussed include: the development of mathematical arguments (argumentation) as part of productive communication in mathematics classrooms, facilitating concept development, reasoning and problem solving (Chapter 5); the understanding of mathematical induction in co-operative setdngs (Chapter G); and a range of conflicts and harmonies amongst emotional, cognitive, and social factors, that can influence mathematical activity (Chapter 7).

Finally, the third theme (Theme III: Practice, Community, Identity, and Politics), consists of chapters which discuss communication and pupils' access to mathematics learning in a broader sociological, cultural, and political context. Topics include: an exploration of how mathematics colonizes pupils' lives in Togo, thereby also giving insights into the colonization of pupils' lives elsewhere in the world (Chapter 8); an examination of the discourse and politics of race and task context in school mathematics as experienced in a boys school in South Africa (Chapter 9); and a model for a pedagogy based on critical communication and addressing the complexities of UK mathematics classrooms (Chapter 10).

We now move into a brief discussion of the concept of communication and then into an overview of current trends and major research studies within each of the three themes mentioned above. A section follows where the focus of the book and the possible "readings" of the collection of chapters are discussed. VW end with a few words about the editorial process.

COMMUNICATION

In a very broad sense, communication is about sharing. Etymologically, the word has its root in the latin verb *communicatv,* which literally means "to make common" or "to share." However, most dictionary definitions are based on the metaphor of information "exchange," "transport" or "transmission" (e.g., Encyclopedia Britannica, which defines communication as "...the exchange of meanings between individuals through a common system of symbols" quoted in Sfard and Kieran, 2001, pp. 40-47). We also witness a current emphasis on the improvement of "communicative skills" as "skills in information exchange" as part of the discourse of an "Information Society" through mainly, but not only, the effective use of information communication technologies in varied sectors *ai* employment, public services, and education (Castells, 1990; Castells et al., 1999).

Anna Sfard makes a strong critique of an oversimplified use of such "folk" definitions of communication for research in mathematics education. She explains that the metaphor of "exchange" implies that information in the form of ideas and feelings can be objectified and moved from one person to another almost un-problematically. In contrast, she argues that constructing meaning is not a private, but a public affair, and thus communication of mathematical meaning is not about exchanging mathematical information, but about co-constructing information and knowledge (see Sfard, 2000, 2001; Sfard & Kieran, 2001). In her view, the need to communicate drives the construction of new mathematical objects and, as such, the process of communication is inseparable from the process of cognition. Further, communication taken broadly is only partly about sharing information with the intention of sharing knowledge or competencies; it primarily implies sharing beliefs, attitudes, values, rituals and behaviors, thereby building and shaping feelings, relationships, identities, communities, practices, ideas, and knowledge. Furthermore, communication presupposes social interaction in one form or another, and thus assumes the resolution or acceptance of any conflicts or differences among participants.

Communication is an integral part of classroom and schooling processes (Mercer, 1995; Edwards & Mercer, 1987). It is now being made more explicit, as it entails a distinctive aspect of educational programs, reflected in curriculum reforms in some countries. A number of the mathematics curricula in elemental")' schools encourage active use of teacher-pupil and pupil-pupil communication (expressed in talking and writing) as peclagogic forms, arid emphasize pupils' development of communicative skills (Cockroft, 1982; NCTM: National Council for Mathematics Teachers, 1989; Undervisningsministeriet: Danish Ministry of Education, 2001). Underlying mese guidelines for curricula!" change is die hypothesis that the quality of communication influences the quality of teaching and learning mathematics (Cobb & Bauersfeld, 1995; Lampert & Cobb, 1998; Pirie, 1998).

Still, how is communication in the mathematics classroom being conceived? Nerida Ellerton and Philip Clarkson (1996) trace the interest in aspects of language and communicauon in die field of mathematics education back to the writings of Biyne fifty years ago. Bryne claimed that, "words are links in the chain of communicauon" and "symbols" of mathematical meanings. They also draw attention to Aiken's review in 1972, which focused on classroom discourse, and to Austin and Howson, who in 1979 presented a framework for discussing classroom communicadon between teachers and learners considering the pardcipants' backgrounds of language, culture and reasoning models (Ellerton & Clarkson, 1996, p. 987).

Recendy, Candia Morgan (2000) and Anna Sfard (2001) both reflect on current and earlier discussions concerning language and communicauon in mathematics classrooms, and observe that research has moved away from a conception of mathematics communication seen only in terms of its specialist vocabulary and symbolism, often subordinated to natural language as exemplified by Ausdn and Howson in 1979. They agree that researchers now place more emphasis on language "in use" rather than on "correct use" of mathematical language. However, Anna Sfard points out diat "...the old infrastructure has already been shaken, but the new foundations are not yet fully shaped" (2001:3). She indicates specific weaknesses in terms of conceptual tools, methodological frameworks, and pedagogical considerations that still need attendon, but at the same time she admits that addressing diese types of needs requires time and collective efforts. We are still at the beginning of what may prove to be a very challenging but worthwhile route.

Candia Morgan further observes that the majority of studies still concentrate on micro analysis of teacher-pupil conversadons in lessons "...as dependent solely on their current social setting ... without considering their histories and futures" (2000, p. 97). The chapters in this book reflect the striving to engage with the social complexities of mathematics classroom communicadon at the levels of classroom, school and community and take research on mathemadcal communicauon a step further. They include studies on how ideologies, values, beliefs, exisdng practices, and expectations influence communicative acdvities.

COMMUNICATION: REGISTER, REPRESENTATIONS, CONTEXT(S)

Communication in mathematics classrooms is ultimately based on understanding and utilizing die symbols, language and rituals of the school mathematics register. In this section, we discuss work which focuses on communication as a question of how a mathematics register develops, as well as work which considers what might be the role of representations and contexts in teaching and learning mathematics.

Teaching and Learning a Mathematics Register

Mathematics education programs have as an integral goal to develop learners' competencies in communicating in a school mathematical language, and using mathematical symbols in what is considered an appropriate manner. Whatever the underlying intention,*-this often boils down to becoming acculturated into a culture of doing school mathematics (reading, speaking, writing, drawing, as well as behaving and valuing), which should not be confused with doing mathematics or using mathematics in relevant out-of-school situations. The task of developing these competences (cf. Niss, 1999) presents one of the main challenges and problems of everyday teaching.

David Pimm studied the mathematical register and its use in mathematics classrooms in the United Kingdom (1987). lie showed that students can be misled by mathematical vocabulary to attach meaning to signs which are not in accordance with the mathematical concepts with which they are intended to be dealing. Sometimes, students create impressions which later turn out to be incomplete or misleading, such as "all numbers are either even or odd" or "multiplication makes bigger."

In *The Nalrnr of Mathematical Knowledge* (1984), Philip Kitcher presented his hypothesis that mathematical knowledge develops through the modification of mathematical practices (which to Kitcher are the practices of mathematicians), consisting of five components: a language, a set of accepted statements, a set. of accepted explanations, a set of questions selected as important, and a set of meta-mathematical views (p. 163). He considers as especially important one particular type of inter-practice transition, that is related to the development of mathematical language. As an example, here only presented in a very simplified form, think of the imaginary unit *iorV—1.* If the referent of "number" is fixed by using (he available paradigms, it restricts the referent to the reals. In that case, *i* fails to refer to anything. If "numbers" are instead diought of as those entities which can be added, subtracted, multiplied and divided, then developing analogies of the ordinary arithmetic operations implies that *i* is a number (pp. 174-176).

As we see it, a parallel can be drawn to the learning of mathematical concepts in school. Heinz Steinbring argues that for mathematical symbols, vocabulary, and other representations to function as signs, they cannot stand by themselves—just like symbols and vocabulary of "natural language," they must refer to a *reference domain* from which they receive meaning, i.e., they have the referent of the sign fixed (2000, p. 82). In other words, they must be associated with some type of experiences. A concept that has been exemplified in various domains and furthermore has the potential to be exemplified in new domains, can develop—as seen with the historical expansion of the concepts of number and function, for instance. On the other hand, the domains in which a concept has been exemplified can resuict the concepts that students construct.

Rudolf voin Hofe has clarified these points in his discussion of "basic ideas" *(Grundvorstellungen* in German) versus "individual images" (1995, 1998). Generally, not one but a number of "basic ideas" underlie a mathematical concept. As an example, the "basic ideas" of multiplication are either (1) repeated addition—putting the same number of things together several times, (2) enlarging something (possibly by a non-integer mea-. sure), or (3) determining the number of possible combinations of a given number of items, vom Hofe uses the term "basic ideas" prescriptively, seeing them as the teacher's tools for the purposes of communication in the classroom. The teacher will specify' the "basic ideas" underlying a given mathematical concept, procedure or result and transpose these basic ideas into a learning context. If, for instance, the mathematical idea with which the teacher wants the students to engage is multiplication of fractions, repeated addition is not a suitable basic idea, whilst enlarging something may be. The teacher looks for a context where "enlarging by a non-integer factor" plays a natural part (e.g., enlarging photos). This context activates students' individual images, their assumptions, and their models of explanation. In mathematics teaching, *individual images* then become a descriptive tool; a notion with which to capture what sense students make, in contrast to what sense the teacher intended diem to make, as was prescribed through the "basic ideas." The students use their individual images to grasp the learning context, which again may lead them to develop "basic ideas" and then eventually (through comparing, reflecting, abstracting, generalizing, synthesizing, etc.) engage with the mathematical concepts.

Obviously, a number of stumbling blocks exist. For one, "inappropriate" individual images can be activated by a learning context, leading to incomplete or inadequate understanding of the basic mathematical ideas that the teacher intended to teach. Second, the teacher may use inappropriate basic ideas, which resuict the development of concepts or lead to a partiallar understanding which may later prove limiting -cf. the examples from Pimm (1987) mentioned above. These stumbling blocks do not only apply to the development of an understanding of mathematical concepts, but also to broader thinking processes such as mathematical reasoning. For instance, Brian RoUnan (1988) has pointed out that a proof is not only a chain of arguments and conclusions expressed in symbolic language; it is based on an underlying principle or narrative which must be grasped in order for the proof to make sense.

Representations, Representational Media and Contextualizations

Representations (concrete, verbal, symbolic, graphic or real-life narratives) are considered important to conceptual development and an essential part of communication in mathematics education classrooms. Their role can be seen as twofold. On the one hand, the teacher can use a number of graphic, iconic, mental or physical representations to communicate ideas about mathematical concepts with students. For example, Claude Janvier has grouped representations of mathematical concepts that could be used both to describe and prescribe classroom activities, such as: situations, verbal descriptions, tables, graphs, and formulae (Janvier, 1987). On the other hand, students themselves can construct their own representations (verbal, iconic or symbolic) or "translations" across representations that become an expression of their understanding and learning progress. For instance, Martin Hughes (1986), Marit Johnsen-Hoines (1987), and Maulfry Worthington and Elizabeth Carruthers (2003) describe how even very young children express in writing their understanding of number and simple arithmetic operations (addition, abstraction, equation) using representations of varied iconic and symbolic types.

Information and communication technologies can provide dynamic media, enabling students and teachers to manipulate and program visual representations of mathematical ideas. Despite the fact that computers and adequate software are not widely accessible to the majority of learners and teachers, we cannot ignore that the development of particular educational software (e.g., Logo, Cabri, Geometry Sketchpad, etc.) or tools (e.g., the graphic calculator) offer new possibilities for teaching and learning mathematics. These possibilities include not only representing mathematical concepis visually (Dubinsky & Tall, 1994), but also linking multiple representations (Kissane, 2002), dynamic manipulation of mathematical objects (Labordc, 1993), active construction of mathematical knowledge and programming (Hoyles *He* .Sutherland, 1989), testing mathematical hypotheses (Andersen, 2003; Mudaly, 2004), informing proof (de Villicrs, forthcoming), critical and reflective use of mathematical information in themed activities or modeling (Christiansen, 1996; Chronaki, 2003), and establishing communities of practice (Andersen, 2003). However, technology is not a factor that by itself enhances (or for that matter hinders) learning. The outcome of using technology depends not only on the type of hardware or software, but ultimately on the role technology develops in close interaction with the roles of teachers and learners within the pedagogic context of classroom communication (Chronaki, 2000b).

This idea is strongly argued by Marcelo Borba, in Chapter 2. He introduces the notions of *technologies of intelligence and humans-with-mediaas* ways of overcoming the dichotomy between humans and technology. He considers how technology together with relevant pedagogies reorganize communication in the classroom. His case study refers to the experiences of a group of biology majors in higher education in Brazil, who use graphic calculators to study the notion of derivative. Borba claims that it is neither the technology, nor the teacher, nor the pedagogy that makes new forms of communication evolve. Instead, the students-with-technology transformed the students' ways of learning and communicating (e.g., having a visual representation of function, making conjectures, checking possible replies, etc). In this sense, more conventional technologies in the classroom, such as "orality" and "writing," are not replaced by the"new technology, but play a collaborative role in the communication process.

The increasing presence of specialized educational media such as hands-on materials, manipulatives and computer programs aims to facilitate the construction and communication of mathematical knowledge. David Pimm (1995) discusses various representational media used for the purposes of introducing or experimenting with concepts in geometry, arithmetic and algebra such as: paper folding, posters, diagrams, abaci, calculators, tiles, geoboards, cuisenaire rods, and the Dienes multi-base arithmetic blocks. Paul Cobb, Erna Yackel and Terry Wood (1992) explain how such external representations are part of the communication emerging amongst teachers and students and rely heavily on the participants' subjective interpretations. As a result, the students do not necessarily see the concepts we intend them to see. For example, Labinowicz (1985), found that many third graders see 600 when they look at the 10x10x10 Dienes' block intended to represent 1000. Cobb, Yackel and Wood make a strong case that students will not discover mathematics simply by dealing with these representations (1992). Lucio Meira (cited in Ainley, 2000), uses the concept of "transparency," as introduced by Jean Lave and Etienne Wegner (1991), to address pupils' difficulties in accessing mathematical knowledge when representational media are in use. Janet Ainley (2000), based on Meira, studies the transparency of graphs and indicates that their meaning in many cases is not easily apprehended by students.

In Chapter 3, Triandafillos Triandafillidis ami Despina Potari, study the integration of written and verbal forms of representations in situations that involve the exploration and construction of models of three-dimensional real objects (i.e., vinegar and soda bottles) by primary school pupils in Greece. They explore the extent to which the integration of different representations can change existing discourses in the mathematics classroom. Triandafillidis and Potari find that, although the new media can challenge the ways pupils communicate their understandings concerning the properties of geometrical objects towards more expressive modes, the acceptance of their choices depends on what teachers value as proper mathematical knowledge. Older students have internalized ibis, and for them, developing mathematical meaning is a complex process of negotiating meaning and task (Christiansen, 1997).

"Reading" mathematics through external representations is not a straightforward process. Sometimes, instead of facilitating the development of concept understanding, they create confusion or even block learning. Gerard Vcrgnaud (1987, p. 229), conceptualizes* the problem of interpreting representations as an interactive "uanslational" play between three entities: the referent, the signified and the signifier (the *ivjetvntis* the real world to be experienced by the learner who acts on it; the *signified* is the cognitive level consisting of internal representations where the learner recognizes invariants, draws inferences, generates actions and makes predictions; the *signifier is* the symbolic system, such as natural and mathematical language, that the learner uses for communication purposes). Richard Lesh, Tom Post and Merlyn Behr (1987), focusing on this interpretation problem, suggest that the competence to interpret mathematics across representations is linked with broader competencies in learning mathematics, such as simplifying, generalizing, describing, acting out, writing, reading, symbolizing, concretizing, and formalizing.

Though not stressed in his chapter, Horba's study also illustrates the effect on the communication and cognitive processes of making modeling activities central in the classroom (cf. Christiansen, 1990). It is one way to link representations that can be expanded into formalizing, symbolizing, abstracting, etc. On the other hand, it implies connecting mathematics with other contexts or practices, which can both facilitate learning and give rise to new problems.

From Representations to Contexts

Recent discussions, from situated cognition theorists (e.g.. Lave, 1988), sociologists of mathematics education (e.g., Dowling, 1998) and poststructuralists (e.g., Walkerdine, 1988), have conceived the problem of interpreting mathematical information in varied situational contexts as based on the wrongful assumption that "transfer" between contexts (or practices) is unproblematic. *Transfer* refers to die use of mathematical knowledge and competencies from one situational context in another (e.g., the application of mathematical knowledge in everyday activities, the use of cross-curricular work, the recontextualizing of out of school themes in pedagogic practices). For example, in her 1988 book,Jean Lave took a strong position when she claimed that "transfer" from "in" to "out" of school practices generally does not occur. Paul Dowling (1998) analyses the employment of "realistic" task contexts in UK mathematics textbooks, and explains how diis can result in a mythologized image of the subject of mathematics as a powerful universal language. Jeff Evans (1999) finds that transfer of learning across situational contexts is not impossible, but it is problematic and undependable. However, based on Valerie Walkerdine (1988), he claims that it is possible to build bridges between such contexts (or practices) through looking for inter-relations between the discourses involved, constructing "chains of meaning" (Evans, 1999, p. 30).

The "reading" of mathematical representations and registers also lies in die cultural contexts through which the readers understand (see Saxe, 1991)—a point captured by Tamsin Meaney in Chapter 4. Resolving the ambiguity of mathematical meanings carried through natural language or symbolic systems is not always a matter of the accurate use of verbal, visual, static or dynamic representations (through media or software). In fact, all representational media seem both to limit and to open up possibilities for die construction of meaning. Representations are rooted in and take meaning from the cultures where natural language (or the formal language of teaching) lies. For instance, .Alan Bishop, in his review of cross-cultural studies, describes languages where generalized numbers do not exist, and where notions about, for example, vertical directions are absent from the vocabulary (1988, pp. 20-59). Vicky Zack also refers to a number of languages not using particular discourse patterns such as "why-because" and "if-then" (1999, p. 134). In contrast, some languages have more meaningful names for geometrical concepts. For example, Kliuzwayo (1999), refers to a discussion with a South African multi-lingual teacher who finds diat concepts such as angle and quadrilateral make more sense to students in Afrikaans than in English due to the more descriptive names.

In Chapter 4, Tamsin Meaney, based on the work of Michael Halliday, adopts a social semiotic perspective, in order to explore the relations between linguistic choices made by participants in a mathematical activity, and the mathematical texts produced. She analyses two contrasting mathematical texts: an e-mail dialogue between two mathematicians, and a classroom discussion amongst a group of students who try to solve a task posed by die teacher. These two situational contexts and the associated conversations differ significantly in terms of the purposes and intentions of the participants in relation to the task, their role and mathematical content, but also in their competencies to use a mathematical register eloquently. Meaney claims that the participants' perceptions are determined by their culture, including their beliefs about mathematics. Her analysis raises important questions for educators working in multicultural and multilingual classrooms. For example: How can existing knowledge be taught without devaluing pupils' self-expression? How can teachers handle teaching the mathematical register and at the same time making explicit how appropriate linguistic choices are made?

In conclusion, we find that the research increasingly has recognized that the cultural context, the natural language and the classroom discourse influence students' reading of symbols, their appropriation of the mathematics register(s), as well as their understanding and construction of concepts and their representations. From focusing on the relations amongst referent, signified, and signifier, the focus has increasingly become what influences these relations. This is .1 move„frgm seeing mathematical language as separate from its use and the context in which it is being used. A mathematical register does not develop in a social vacuum. It is used alongside the everyday registers (in either mother-tongue or an official language) of teachers and learners. In this sense, a mathematical register develops along with metaphoric and metonymic expressions, reference maicrial, and representations thai reflect (and are rooted within) the culture and discourse of a particular group of people.

Although neither the "translation" between concrete and symbolic experiences, nor the transfer across situational contexts are easy processes, the use of multiple representations and contexts in mathematics classrooms not only may further desired learning; they are unavoidable. However, though the use of "realistic" themes in textbooks, assessment items, or lesson planning signifies a desire to present mathematics as friendly and close to real life, it ma}' also cause a gap l>etween the context of the "theme" and the mathematical activity itself (Christiansen, 1990; Chronaki, 2000a). Discussing how to address Uiis problem, Jeff Evans (1999) suggests the use of context-sensing (or contexung) questions as a tool for unraveling different aspects of the contexts involved and bridging the related discourses.

Viewing mathematical learning through register, representations, and contextualizations has received quite a lot of attention in mathematics education, providing us with insights in how signs further meaning making, how signs limit the intended meaning making, and how the development of meaning and of symbolizing is intertwined with classroom communication. It has provided us with insights into the role of signs in prevalent forms of classroom communication, as well as in classrooms where the communication has been altered with the specific purpose of exploring possible changes. And over the past decade, it has provided us with valuable insights in the problems of using realistic task context or expecting transfer across situational contexts.

In our view, the progression towards embracing knowledge as situated, yet acknowledging the possibility of exploiting overlaps in practices or linking situations through constructing chains of meaning between discourses, has taken this area forward by a giant leap. It embraces learners as whole people, with unbreakable links amongst emotions, body, intellect, social belonging and spiritual feeling. Within this school of thought, with its focus on the unitary self of the individual learners, we would like to see this further utilized in discussions of learning and communicating the mathemadcal register, in linking signifier, signified and referent, in determining what are appropriate communicative contexts through which "basic ideas" can be shared, and in understanding learners' own mathematical images.

COMMUNICATION: SOCIAL INTERACTIONS, SOCIAL SETTING, CLASSROOM ACTIVITY

The process of developing mathematical communication takes place in a social setting that involves acdvity, including, interaction amongst human beings and human interacdon with tools (i.e., conceptual, material, historical). Although the "social" is regarded important within many schools of research in mamemadcs educadon, the connecdon between social interaction, communication and learning has been approached in rather different ways (see de Abreu, 2000; Sierpinska, 1998; Cobb & Bauersfeld, 1995 for more details concerning conceptual and methodological differences amongst various frameworks). Along with exploring the role of language, die organization of classroom communication has been analyzed through die lenses of the organization of interactions. Among the issues being researched are: how people work and interact in groups, what is the influence of their interacdon on learning mathematical concepts, what is the influence of an expert, what is the role of language and register in the interacdon, what patterns of talk and norms of behavior emerge, and even what alternative patterns can be encouraged. We will briefly outline some dominant schools of drought, such as groups that adhere to construcdvist, interactionist and socio-cultural perspectives.

Social Interactions and Individual Learning (Constructivist Perspectives)

Researchers of the Geneva group, or the *neo-Piagetians* (Perret-Clermont et al., 1991) base their work on Piagct's view that "... social interaction is necessary for the development of logic, reflexivity, and self-awareness" (quoted in Cobb & Bauersfeld, 1995, p. G). Unlike Vygotsky, who claimed the primacy of the social over individual learning, they argue that interpersonal interaction triggers socio-cognitive conflict between learners, which, in turn, gives rise to individual cognitive conflict. As learners try to resolve these conflicts, they become aware of their activity and construct increasingly sophisticated systems of thought.

Radical constructivism is also grounded in Piaget's view of learning as a "self-organization" process based on assimilation, accommodation and search for viability through interaction with others. It has been advanced by the work of Ernst von Clasersfeld, who adheres to the premise that truth (and language use) is always subjective and related to the effective or viable organization of one's own activity (von Clasersfeld, 1995). Communication is dins seen as a complex process of mutual adaptation wherein individuals negotiate meanings by continually modifying (heir interpretations.

The *Purdue Problem-Centered Mathematics Pivject* by Paul Cobb, Grayson Wheatley, Terry Wood and Erna Yackel, follows the above ideas, exploring mathematics teaching and learning in classrooms organized in so-called *inquiry mathematics teaching* (Cobb, 1991, p. 13). Inquiry mathematics teaching is characterized by specific communicative patterns that encourage students' active construction and negotiation of mathematical meaning (see Cobb, 1995 for an extensive discussion). In their research, adhering to what is known as a *socio-consttwtivist* approach, they take as units of analysis pupils' work in small groups and teacher intervention. A basic premise is that interaction and communication among pupils or pupils and teachers are seen as creating opportunities for negotiating meanings and developing taken-to-be-shared understandings (Cobb, 1995). They are interested in how socio-mathematical norms are constituted and stabilized in the mathematics classroom and how they promote learning. The term *sneio-mathematical norms* describes a set of values with regard to mathematical activities in classrooms, for example what counts as insightful or elegant mathematical solutions, what might be seen as acceptable ways of talking, writing, interacting etc. (see also Kitcher's description of a mathematical practice).

Their work has led to a focus on the development of a *reflective discourse* in mathematics classrooms. Paul Cobb, Kay McClain and Joy Whitenack write that through reflective discourse, "...mathematical activity is objectified and becomes an explicit topic of conversation" (1997, p. 258). Although "reflective abstraction" is an individual process, reflective discourse is present in the social processes in the classroom, and, as they suggest, appears to support students' reification of their mathematical activity (Christiansen, Nielsen and Skovsmose, 1997 discuss reflections and reflective discourse in another sense, namely reflecuons on applications of mathematics and on students' perceptions of learning mathematics.) Paul Cobb, summarizing much of the work done in the team, also links this to the development of appropriate forms of notation: "In general, we have found that the development of classroom discourse and the development of ways of symbolizing and notating go hand in hand and are almost inseparable" (cited in Sfard et al., 1999, p. 47). On this basis, they argue for inquiiy mathematics teaching, as an alternative to currently dominant practices.

The above approaches, rooted in Piagetian theory though with some influence from other schools of thought, use mainly a psychological analysis for investigating the individual's subjective interpretations. As such, their potential to conceptualize and explore,the role of the "social" in classroom mathematical activity has been questioned. For example, Stephen Lerman (2000) argues that the study of the "social," including language use and communication, takes a secondary place in studies such as these. And although Solomon (1989) agrees that social interaction can raise cognitive conflict, she claims that pupils' solutions are not always compatible with "the taken-as-shared" mathematical symbols and practices of the wider community. In other words, she challenges the view that "negotiation of meaning" and "symmetrical relations" can ever become norms of classroom communication and learning. Instead, along with Walkerdine (1988), Solomon claims that what counts as a mathematical problem and as a solution is deeply social and that the roles taken by teachers, pupils and curricula always entail "power" and "control" over what is being learned and communicated. However, recent advances in the Geneva group, known for its neo-Piagetian or psycho-social approach (de Abreu, 2000, p. 2), involve the coordination of traditional psychological methods with anthropological ones and exhibit an interest in investigating the role of the social more deeply. The aim is to explore how the macro and micro-social context(s) interact and influence pupils' interaction and learning through "...die evocation of social experience by symbolic means" (de Abreu, 2000, p. 13). Also, a collaboration between the Purdue Group and the Bielefeld Group (Heinrich Bauersfeld, Gotz Krummheuer & Jorg Voigt) tries to co-ordinate a psychological and an interactionist analysis of the classroom micro-culture, arguing that the individuals' positioning and die classroom culture are reflexively constituted (Cobb *8c* Bauersfeld, 1995).

Patterns of Interaction in Classroom Communication (an Interactionist Analysis)

From the *interactionist* perspective ascribed to by the Bielefeld group, communication is perceived as a process of often implicit negotiations in which subtle shifts and slides of meaning frequently occur outside the participants' awareness (see Krummheuer and Voigt, 1991; Cobb & Bauersfeld, 1995). For the study of social interactions and communication, the local classroom micro-culture is taken as the reference point. Gotz Krummheuer and Jorg Voigt (1991) explain that the work of their team at Bielefeld constitutes an evolution from earlier work in mathematics education, which focused either rather narrowly on content or on the psychology of learning, to a focus on interactions in the classroom, manifested in the extensive use *of interaction analysis* of everyday school situations. They summarize their theoretical constructs this way: "In the social interaction in mathematics instruction, pupils and teacher interpret educational objects and processes on the basis of different subjective realms of experience and within different frames" (p. 18, our,translation), hi their work, the notion of frames refers to the habitual patterns of interpretation:

> These frames assimilate through modulation, without necessarily overlapping. A provisional work "space" (or a working interim) thus is produced in the process of developing understanding. Its conflict potential is disarmed through routines. Through a "train of compulsion" in the social interaction, the routines are linked together into patterns of interaction, (p. 18, our translation)

The aim is mainly to describe patterns of communication and interactions *per seas* they occur in the classroom culture and not to prescribe how these should be organized. Sierpinska explains that for the interactionist, communication is not meant in a transitive form; "Meanings are not in people's heads to be transmitted from one person to another. People do not have to mean what they say, but what they say definitely means something, not just to 'others' but something in the given culture" (1998, p. 52). The role of communication is seen as fundamental and as a prerequisite for language/register acquisition. Heinrich Bauersfeld (1995) conceives communication as a series of "language games" specific to mathematics classrooms, generally characterized by the competence in what he calls *technical "langnaging.* "He cites Ludwig Wittgenstein, who writes that "[l]he term 'language game' is meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life" (quoted in Bauersfeld, 1995, p. 279). Further, he prefers using the term "languaging" instead of "language use," because it makes explicit the relationship between language games, classroom micro-culture, and consideration of individual differences. The processes of talking, writing, drawing or constructing can be seen as an accomplishment of "language games" and "languaging," performed by teachers and students in their attempts to negotiate "taken-as-shared" meanings and signs of mathematical activity. As examples of encouraging a flexible and multidimensional "languaging," he mentions that teachers can offer "open" tasks, whole class discussions, ample chances for students to demonstrate their own ways of thinking, and an early introduction to the process of argumentation.

Other contributions of this group involve an analytic ethnography of argumentation (Krummheuer, 1995) and a detailed exploration of thematic patterns of interaction (Voigt, 1995). Gotz Krummheuer sees argumentation, not only in the Aristotelian sense of rhetoric (i.e., a process accomplished by a single person confronted with an audience that is to be convinced), but as a basic communicative aspect of everyday activities, offered also in classrooms, such as arguing, explaining, justifying, illustrating, exemplifying and analogizing. This view is also elaborated by Nadia Douek in Chapter 5. JorgVoigt has offered a detailed analysis of patterns of interaction in mathematics classrooms. He explains that communication and especially "negotiation of meanings" is fragile, and there is always a risk of disorganization. He argues, thus, that patterns of interaction function to minimize this risk. Voigt (1985) compares two different patterns, the *elicitation pattern* and the *discussion pattern.* In the elicitation pattern, the students offer an answer and solution to a problem posed by the teacher, and the teacher evaluates or guides students towards a definite argument, solution or answer. In the discussion pattern, on the other hand, the teacher asks students to report on how they arrived at their answers and solutions, and then the teacher contributes widi further questions, hints, reformulations or judgments.

Along similar lines, Terr}' Wood (1995) discusses *the funnel pattern* and *focus pattern.* In the funnel pattern, the teacher uses a series of guiding questions that narrow students' queries until they arrive at the correct answer. In contrast, die essential aspects for solving a problem are brought to the fore in the focus pattern, where the teacher's role is to indicate what are the critical features of the problem, to avoid providing the solution, and to summarize important parts that lead to the solution. Voigt and Wood seem to suggest that patterns such as "discussion" and "focus," tend to promote specific communicative norms in the mathematics classroom which characterize an "inquiry model of mathematics teaching," mentioned earlier (see Cobb & Bauersfeld, 1995; cf. Christiansen with J01gensen & Geldmann, 2000 for an example from mathematics teacher education).

Further, Heinz Steinbring (2000) supports an interactionist analysis with a strong epistemological basis, arguing for the reciprocity of social internetions and the epistemological constraints of mathematics (its signs, symbols, the language means etc.). He explains that:

> ...Only when mathematical symbols and signs are interpreted as intentionally expressing relationships and structural connections, mathematical communication could become a vivid social process in which all partners have to construct their mathematical understanding by actively interpreting the signifiers conveyed by other communication partners. (Steinbring, 2000, pp. 87-88)

Consistent with this view, the work by the Purdue group has developed the notion of socio-mathcmatical norms, the Bielefeld group has coined the term "thematic patterns," by which interactions that involve mathematical activity are analyzed, and the socio-culturalists base their analysis on Vygotsky's conception of how scientific concepts are developed. These works all attempt to co-ordinate the social and tire individual in their analyses without ignoring the development of disciplinaiy knowledge.

The Primacy of the Social Setting in Learning (a Socio-Cultural Approach)

Whereas theorists from the interaction is t perspective propose that individual students' mathematical activity, the curriculum, and the classroom microeulture are reflexively constituted, those who work within the socio-cuhuralist perspective, argue that the "social setting" has a primary importance over what is learned and how it is learned (see de Abreu, 2000; Lerman, 2000). This premise is based on Vygotsky's claim that all intellectual development evolves from the interpersonal (i.e., interacting with others in a historical, cultural and political context) to the intrapersonal (i.e., engaging in subjective actions and thinking); in other words, from the social domain to the individual (Vygotsky, 1978, 1986; Wertsch, 1985). Vygotsky says:

> Any function in the child's cultural development appears twice, or on two planes. First, it appears in the social plane, and then on the psychological plane. First it appears between people as an mterpsychological category, and then within the child as an intrapsychologieal category, (quoted in YVcrtsrh & Stone, 1985, p. 164)

Communication is realized via a complex net of interactions in the social setting and is regarded as an important route for intellectual development and for the historical constitution of human consciousness. Alexei Leont'ev argued that thought develops through purposeful activity, and in particular activity that is oriented around some practical and olyect-specific goal (1981). A number of theoretical constructs such as the notions of *cultural tool, mediation*, or *zone of proximal development* consist of key analytic tools for explaining aspects of development and instruction.

A series of "cultural tools" can have either a material, conceptual or historical nature and can contribute to die social construction of knowledge through becoming thinking or psychological tools in die activity of problem solving (Mellin-Olsen, 1987), Maria Bartolini-IJussi (1996) and Anna Chronaki (1998) have analyzed what might be the nature of such "tools" in situations involving die teaching and learning of perspective drawing or geometric transformations in primary or lower secondary classrooms. They identify "tools" such as the use of photographs, acquaintance with methods of real life drawing used by artists or architects, construction of classifications for what changes and what not in perspective drawings or in geometric movement, application of specific mediods for construction, etc. As a result, they argue that the semiotic "mediation" of tools is realized in the classroom in manifold ways (e.g., a tool developed elsewhere is drawn into new problems, a new tool is created by pupils, old tools can be introduced through the introduction of historical examples of methods, or new tools can be suggested as means for focusing observation and thinking). Christiansen (1996) discusses realistic modeling as a "tool" in problem solving in high school mathematics, and what it means to students' construction of knowledge and interactions in peer groups and with the teacher.

The *zone of proximal development* (ZPD) was developed by Vygotsky in the 1930s as a means to examine the potential of instruction in purposefully organized educational practices. He defines the zone as "...the distance between die actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). The role of an "expert" is dius becoming the focus of attention to discuss issues such as: how semiotic mediation forms internalized self-regulative speech (Bruner, 1985), how an adult and a child who take different perspectives in communication can interact in specific situations (Hundeide, 1985), and how the ZPD becomes a general mechanism through which culture and cognition interact (Cole, 1985). Language use, tool use, motives, actions and operations within the boundaries of the activity are thus seen as central units for the analysis of communication and interactions, exploring in depth the values of motives implicit in any activity. According to Benjamin Lee, Vygotsky has argued that the development of higher forms of thinking, such as scientific thought, demands that the individual "...differentiatfe] the various levels of generality and communication in his language" and that "...communication and generalization are inextricably linked" in the formation of scientific concepts and thinking (Lee, 1985, p. 7).

According to Nadia Dotiek in Chapter 5, the teacher plays a crucial mediating role in organizing and orchestrating pupils' activity in order to facilitate argumentation that will support conceptualization. She proposes a number of classroom communication practices that encourage this interaction, and can be organized by the teacher, such as: individual oral argumentation with the teacher, collective classroom discussion managed by the teacher, and individual written argumentation as reaction lo other students' texts. Argumentation is considered as a particular development of communication that presupposes a "contradictory listener," the development of consciousness, and the organization of concepts into systems. Although she does not develop the argument in this chapter, Douek also claims that argumentative communication in classrooms facilitates competencies in mathematical proving.

This topic is taken up by Inger Wistedt and Gudrun Brattstrom in Chapter *6.* They problemalize classroom communication, exploring die merits and limitations of peer discussion and < ooperadve work in developing the competency to prove "by induction through investigating the particular case of non-algebraic tasks. They argue that active 'participation alone does not safeguard that all students will grasp the cultural and cognitive practice of their activity. Wistedt and Brattstrom argue that such knowledge rests on a meia-theorctical level and this type of knowledge allows students access to the practice that enables them to understand and change their views of what constitutes mathematical knowledge within a particular domain. Wistedt and Brattstrom observe that students as new-comers in a practice cannot access this level of communication, and, as a result, they cannot coiitextualize their discursive activity, resolve conflicts or understand the meaning of particular mathematical activities. Thus, they stress the importance of a more experienced member who interacts with students to enable a challenge to pre-existing assumptions and to encourage inclusion of what, to students, are alternative ways of proving—in this case by means of geomeuic reasoning.

When acknowledging the primacy of social interactions in the learning process, we cannot ignore the influence of interpersonal relationships, particularly the emotional aspects. Chapter 7, by Dave Hewitt, highlights the communication and learning that takes place in the interplay of social, cognitive and affective factors rooted in the relations between individuals. Through various classroom examples, he illustrates how these issues are sometimes in harmony and other times in conflict, and he rellects on when cognitive, social and emotional aspects work for or against mathematical learning. Hewitt talks from a teacher's pedagogic: perspective about the existence of all three aspects, and he stresses the conflicts and/or harmonies that can occur. His analysis does not adhere strictly to any of the above perspectives, yet demonstrates the value of pedagogic descriptions and acknowledges the importance of affective factors.

Valerie Walkerdine (1988, 1998) in her seminal work, has argued about the important relation between subject positioning in the school practice and his or her cognitive development. Through examples, she discusses how classroom activity is being co-constructed amongst learners and educators in a complex interplay of varied discourses that bring to the foreground cultural and linguistic conflicts. As a result, the process of learning is "emotionally charged" as Jeff Evans (1999) argues and involves not only ideas and strategies but also values and feelings. The cognitive and the affective are linked, and as such the potential for "learning transfer" depends on relationships of signification amongst contexts, discourses and practices. Anna Sfard and Carolyn Kieran (2001) also argue against the split between cognition and affect and explain the essential roles of a meta-discursive dimension (e.g., emotions, values, feelings, intentions, motives, or interests) and a discursive focus in mathematical communication.

The work discussed above implied that this recognition of students as individuals with dreams, hopes, plans, etc., is largely ignored. This goes beyond saying that students have sentiments and that their learningis both cognitive and affective. It fundamentally is about the recognition of students as subjects. In that sense, it relates directly to considerations of students' control over various aspects of the educational situation, and its potential to lead to double binds for the students, as also discussed by Stieg Mellin-Olsen (1991).

The Individual versus the Social Debate

In this section so far, we have discussed three perspectives that have provided useful studies concerning classroom communication and learning. Anna Sicrpinska (1998) has provided a similar categorization in her discussion of dominant theoretical frameworks, but we differ in order. She starts discussing constructivism, then socio-culturalism and ends with interactionism, suggesting that this last perspective provides a more promising direction for research on language and communication. Although we also start with constructivism, agreeing that studies in this theory put less emphasis on communication, we keep interactionism in the middle and end with socioculturalism. However, our aim, with this reverse order, is not to suggest the primacy of one perspective over another. Instead, we want to emphasize the value of each approach and the need for more collective work.

We have also outlined how "social interactions" play a different role in communication in these three theoretical perspectives. On the one hand, (he neo-Piagetians, the socio-consiruc tivists and the interactionists are concerned with the analysis of classroom communication and view "social interaction" as the plane of verbal and written discourse. For them, social interaction plays either a secondary or a reflexive role in relation to the individual's actions. Communication is seen more as a process of negotiations of meanings amongst individuals. The socio-culturalists, on die other hand, argue for the primacy of social interaction (both in the historical and collective sense) in the development of learning and its crucial role in organizing productive communicative situations. The "social" is conceived in its human, material or historical dimension and can be realized through the semiotic mediation of more knowledgeable others, peers or tools. Communication and language is seen as a major channel for the constitution of meaning which does not depend solely on individuals' ability to negotiate meanings but mainly on the availability and access to historical, cultural and cognitive tools, as well as the potential for interaction with more knowledgeable others.

The emphasis on the "social" versus the "individual" debate as a key issue characterizing differences amongst these perspectives has dominated current discussions. While this has provided useful thinking and constructive work within frameworks, it in our view also presents a trap for mathematics education research for two reasons. First, even if one was unly primary over the other, it seems to be impossible to determine which that is. The chicken or the egg ... Thus, pragmatism warns us to act with caution in assuming the one framework before the other. Second, the discussions seem to insist on the correctness of one framework at the expense of the other. This excludes (he possibility of seeing the frameworks as complementary. In our view, the blind spot of the one framework is where the other framework resides. Let us make a parallel to physics: the electron behaves as a wave in certain circumstances and as a particle in odiers. It cannot be seen as both at the same time; the descriptions "wave" and "pardele" exclude each other. This forces us 10 see the electron as something in its own right which cannot be comprehended fully by either description but which has characteristics that in our present framework are mutually exclusive. This makes it clear that "wave" and "particle" arc only metaphors for aspects of die electron. In a similar way, we can sec the social as primary or see the individual as primary; these are complementary descriptors, and it. is often worthwhile to consider what assuming the one or the other will imply, rather (ban taking one for grained.

According to Valerie Walkerdine (1988, 1998), classroom communication situations bring to the foreground varied discourses thai reflect the parlicipanis' c uliural and linguistic backgrounds, personal histories, positioning, and emotions. Communication is siiuated in a much wider context than the classroom walls—it is not only what teachers, pupils and textbooks represent, but also the discourses, ideologies, and values that work unconsciously and influence their actions, utterances, motives, and evaluations. Thus, organizing the process of communication in order to facilitate desired learning requires much more Uian the careful organization of social interacdons. It requires embracing learning as involving aspects of identity formation, belonging to communides, and acdng with a historical, social and political context. As pointed out in the previous section, the work within this perspective could benefit from considering die goals and motives of students, in line with *Activity Theory.* While the group of research contribudons described in diis section has expanded the perspectives of the first group, they still largely assume an a-historical perspective of classroom interactions, ignoring aspects of subject posidoning in terms of cultural and social capital. As the silences of the educadonal reality are increasingly being voiced, this dimension has come to play a bigger role in mathematics education research. Methodologically, this also calls for an approach that draws in ediical values as a potential source of evidence (cf. William, 1998). These aspects are considered as equally important elements in a pracdce perspecdve on learning, and this will be discussed in detail in the next section.

COMMUNICATION: PRACTICE, COMMUNITY, IDENTITY, POLITICS

In mis section, we consider communication from the perspective of social or sociological theories of learning, taking into account the nature of the practices in which communication evolves, and which at the same time give structure and meaning to all the activities involved, including communication. Whereas Shirato and Yell (2000) see communication as a practice in its own right, "the practice of producing meaning," the way we use "practice" in this section is to refer to activity imbued with structure and meaning through its historical and social context. Etienne Wenger argues that "practice" denotes much more than being engaged in activity; it is activity in a community diat shares a repertoire reflecting the history of the particular community of practice:

> The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do. f n this sense, practice is always social practice. Such a concept of practice ... includes the language, tools, documents, images, symbols, well-defined roles, specified criteria, codified procedures, regulations, and contracts that various practices make explicit for a variety of purposes. But it also includes all the implicit relations, tacit conventions, subtle cues, untold rules of thumb, recognisable intuitions, specific perceptions, well-tuned sensitivi-

ties, embodied understandings, underlying assumptions, and shared world views. (Wenger, 1998, p. -17)

In the following sub-sections, communication is discussed as an integral aspect of practices in which the communities of learners, their developing identities, and the politics involved are all considered. Exploring communication in mathematics classrooms through these notions involves considering the aspects of developing mathematical registers and organizing social interactions but Uiese are seen as inseparable from a wider context of situated activities.

Practices of Mathematical Activity

It is not a new idea in the philosophy of mathematics to focus on the practice of mathematics rather than on mathematical knowledge alone. For example, Ludwig Wittgenstein (1953) pointed out:

> Of course, in one sense mathematics is a branch of knowledge—but still it is also an activity [...] It is what human beings say that is true and false: and they agree in the language (hey use. That is not agreement in opinions but in form of life, (quoted in Ernest, 1991, p. 31)

The discipline of mathematics is lull of tacit conventions and rules of thumb that are often not formulated in clear logical forms but expressed and accepted through social procedures within the professional communities of mathematicians. Alan Bishop (1988) has explored the practice of mathematicians and users of mathematics in different cultural sites, described through the prevalent values of these practices. His work illustrates just to what an extent mathematical practices of all sorts arc imbued with cultural values. Leone Burton and Candia Morgan (2000) have identified different styles in the ways mathematicians express and justify mathematical arguments in their texts. They explain that the mathematical practices are reciprocally related to their ways of using mathematical language and their conceptions about mathematical knowledge and learning.

The practices of professional mathematicians, are not the same as the practices of school mathematics, and both differ from the practices of applying mathematical knowledge in everyday practices such as shopping and cooking or in technological practices within science, engineering, social science or the arts. All these practices produce distinct discourses. Tamsin Meaney, in Chapter 4, compares the communication amongst mathematicians and amongst school pupils working on particular mathematical problems, and reviewing her examples from the perspective outlined above, we find many of Wenger's points exemplified. It is clear how the repertoires of the two practices are distinctly separate; we see the duality of participation and reification in making meaning; we recognize how the repertoires shape the negotiation of meaning taking place, and yet they are still ambiguous enough to require such negotiation to take place in order for the participants to experience meaning. The practices differ in the type of reificadons with which they engage. The mathematicians engage with mathemadcal concepts and symbols as if they were existing objects, while the students engage with the written task text, attempting to decode it to give direction to their activity. Thus, the social enterprises have different purposes; in *Activity Theory* terms, they are different acdvities entirely.

The students cannot apprehend the pracdce of their mathematics classroom and its social interactions without taking into account the mathematical meaning they derive from their experiences in this context (Steinbring, 2000), but neither can they develop mathematical meaning from experiences in the classroom without apprehending the patterns of participation in die mathematics class. In line with this, Paul Cobb finds that: "...in helping students explain their task interpretations, the teacher is simultaneously initiating and guiding the renegotiation of the socio-mathematical norm of what counts as an acceptable explanation" (quoted in Sfard et al., 1999, p. 47). Through this negodadon, die belonging to a classroom community is also cemented, but yet the quesdon of what diis "negodadon" really means to students and how it takes place remains open.

The connecdons between school mathematical knowledge and various forms of everyday practices (e.g., shopping, street vendors, etc.) as well as meir re-contextualization in pedagogic discourses (e.g., textbooks, tests, etc.) have been analyzed by a number of researchers (Cooper & Dunne, 2000; Dowling, 1998; Lave, 1988; Nunes, Schliemann & Carraher, 1993). They identify that although the tasks appear to require similar madiematical activity, there were major differences in the approach taken in school and everyday settings, respecdvely. For instance, Jean Lave studied how adults use mathematics in supermarket shopping activities such as deciding on the best buy, and found that the shoppers rarely used mathematical operadons learned in school. She highlights that the practices of school mathematics frame a very particular social learning context (Lave, 1988). Paul Dowling (1998) and Cooper and Dunne (2000) point out how the recontextualizing of mathematical applications from everyday contexts into school contexts (e.g., via textbooks or test items) serves to differentiate among esoteric mathematical knowledge and knowledge required for making sense of the application of mathematics in the realistic context. In particular, pupils from lower working class backgrounds have difficulties identifying the required mathematical aspect of such tasks and cannot link the produced discourses of mathematics to the realistic contexts.

Generally, the practices of school subjects play a significant role in shaping participants' perceptions of the actions required in a given activity. For example, Jan Wyndhamn (1993) found that when pupils were asked to solve the same task (putting stamps on a letter), they did unnecessary calculations in the mathematics classroom but not in the social sciences classroom. However, we need to stress that there is not only one practice of school mathematics. Different practices can be identified when comparisons are made across countries, across school communities, and across (as well as within) classrooms. Often, there exist a number of parallel practices within the classroom itself, among the students in particular—practices where what counts may be the color of one's new pencil or the ways in which the official curriculum is resisted (Aim & Skovsmose, 2003). So while students are involved in becoming part of a community of mathematics learners, they are also forming their identity as mathematics learners, which can vary across aspecti uih from the good student to the rebel who is part of a community resisting, the submission to schooling. These choices of participation and identity often reflect the conflicts between the school discourse and broader socio-political discourses to which the students belong such as class, race and gender.

In Chapter 8, Wenda Batichspies narrates the life of students who resist the mathematical practices offered in mathematics classrooms in Togo, West Africa. She discusses the meaning of mathematics and communication in mathematics classrooms for those students and teachers who live within a complex set of conflicting experiences from a French colonial educational system and African contexts. Taking a sociological perspective, her analysis brings to the foreground implicit repertoires and negotiation of meanings constituting communication in the Togolese classroom. The teacher as the primary actor in the classroom teaches "the practice of numbers," while students play a secondary, silent role, and respond mainly when asked. But, at the same time they develop their own communicative and "resistance" codes (e.g., being silent towards the teacher but talkative and cooperative with classmates). More than giving insights into the power, cultural and social relations in Togolese mathematics classrooms, Batichspies' explorations allows us to look at classrooms in our home countries and see the colonization of our students through the imposition of mathematical ideas upon them. For example: the lack of "higher level" mathematical thinking, the emphasis on rote skills, the disciplining of the body so that to engage in the practice (e.g., sharing shoes in order to create an image of a proper student, or manipulating certain situations in order to create personal advantages).

As we outline in the next section, die re-shaping of identity has been addressed in the socio-political theory of Basil Bernstein. For immigrant students, or students who study in a language different from their own, tiiis is reflected not only in the dominant language but also through different practices in their new mathematics classroom that can provide stumbling blocks to participation and negotiation of meaning (de Abreu, Bishop & Presmeg, 2001; Adler, 2001; Gorgorio *8c* Planas, 2000).

Identity and Political Dimensions in Pedagogic Practices

Practice, and in particular "pedagogic practice" is used by Basil Bernstein as a way to talk about learning not only within the school, but also learning in other arenas such as doctors and patients, architects and planners, etc. In this sense, his notion of practice also involves learning as doing, learning as becoming part of a particular community, learning as negotiating meaning, and learning as producing and reproducing identities. He explains: "...the notion of pedagogic practice which I shall be using will regard pedagogic practice as a fundamental social context through which cultural reproduction-production takes place" (Bernstein, 2000, p. 3).

Bernstein approaches the cultural reproductidn-production (including gender, race and social class), through considering the structure and logic of the pedagogic discourses, and the forms of communication amongst agents involved in the practices. Bernstein (2000, p. 4) views this process as highly political and asks: "...how does power and control translate into principles of communication and how do these principles of communication differentially regulate forms of consciousness with respect to their reproduction and the possibilities of change"? Bernstein analyzes "pedagogic practice" in the school context through the components of curriculum, pedagogy and evaluation. Students' access into these pedagogical practices is argued to be influenced not by an inert cognitive ability, but primarily by their social and cultural capital that determines their subject positioning (cf. also Walkerdine, 1988).

Bernstein (1990, 2000) has developed a detailed theory linking a series of concepts. These include: classification and framing (applied in exploring power and control relations amongst knowledge, pupils and teacher), contextualization and recontextualization (applied in exploring how disciplinary knowledge becomes part of pedagogic discourses in the form of tasks, textbooks, test items), and the rules of discursive order such as instructional and regulative (or socially based) discourse. Together, these concepts constitute a *language of description* that enables a detailed analysis and interpretation of how relations of power and control through interactions and communication unfold in pedagogic practices. Bernstein's perspective has recently influenced the work of a number of researchers in the field of mathematics education who are interested in highlighting the socio-political context of mathematics learning in school communities (see

Lerman and Zevenbergen [in press] for an outline of how the theory is currently used in mathematics education research). The notion of "practice," in the ways exemplified by Wenger or Bernstein, helps us to appreciate the deeply social, historical and political nature of classroom communication and learning. Wenger (1998) continues this line of thinking by intricately linking identity formation, development of meaning, practice participation, and the social aspects of belonging to a community—all seen as aspects of learning.

As much as the community component of learning has to do with belonging, it has to do with becoming. In the process of coming to belong to a particular community, the identity of the agent is also shaped. Wenger (1998, p. 5) sees identity as "a way of talking about how learning changes who we are and creates personal histories of becoming in the context of our communities," and he states that:

> Building an identity consists of negotiating ihe meanings .of our experience of membership in social communities. [...) Talking about identity in social terms is not denying individuality but viewing the very definition of individuality as something dial is part of the practices of specific communities, (pp. 145-146)

In the mathematics classroom, students may develop identities such as "the troublemaker" or the "disciplined," "the genius" or "the slow learner," and so forth. These identities may hold one value attached by the teacher and another by fellow students. Chronaki (in press), based on (he study of two ethnic minority girls through their interaction with an adult student teacher on school arithmetic tasks, discusses how the educator's expectations and the students' positioning influence how their identities can potentially be shaped and developed into "learning identities."

Dalene Swanson, in Chapter 9, explores how a group of black students in South Africa experience mathematics instruction and reconstruct their identities as learners in a new school context. These students, through scholarships, are "given" access to participate in the mathematics classrooms of an affluent historically white school. Their move from the home school to the new environment means a degrading in the evaluation of their mathematics competencies. They are positioned in a lower stream of mathematics classes due to their lack of fluency in English and their supposed lack of experience, which could inform their participation in (he higher stream classes. As a result, their opportunities for full participation in core mathematical activities are limited, and their identities in some respect are cemented as "lacking." This strongly reflects the predominant view on black students in the South African Apartheid education system (Kliuzwayo, 1999). The agency of school mathematics and the role of the wider social and political context (regulative discourse) in establishing subjectivity within the classroom discourse are pivotal to this discussion. Swanson argues that the main reason that these children remain disadvantaged is their lack of awareness about the communication gap created between teachers, students and pedagogic practice. Students often do not understand what they are required to do in particular test items or tasks, and as a result they cannot produce a successful reply.

According to Swanson, the main reasons do not reside in students' ability to handle language, but in die pedagogic discourse as expressed through the use of madiematical content, materials, test items as well as the more subtle regulations of participation in classroom practice. All these may or may not allow students access to the regulating discourse of the school mathematics practice, and thus influence whether diey succeed or fail. In that sense, teachers' expectations, informed by their perceptions of students' backgrounds, strongly influence their interaction with students. In the process, the identity element of a student concerning his or her mathematical ability is constructed, reflected in performances and participation. This is similar to R.P. McDermott-(1993), who claims that the reified category "learning disability" acquires a certain proportion of children, as long as this categorization is given life in the organization of tasks, skills, and evaluations of schools.

Belonging and Becoming as Integral Elements of Communication

Communication is an essential part of the practices of mathematics classrooms. The "practice" perspective enables us to relate communication with the root ideas of belonging (participating in a community) and becoming (developing an identity). Bernstein's theory promotes the analysis and clarification of the social and political character of such processes. He argues that having access to the rules of a practice (e.g., the mathematics classroom activity) or being able to develop a corresponding identity (e.g., the successful learner of mathematics) are highly influenced not only by what happens in the teaching context (instructional discourse) but also by the discourse(s) played at the school and community level (regulative discourse). As Swanson, in Chapter 9, claims:

> Classroom mathematics communication is, therefore, not merely about 'transmission' of mathematical ideas within a neutral context, or a conduit of 'mathematical language' outside of school mathematics. Rather, it is a set of activities, interactions or practices which are socio-culturally and politically

situated and serve to produce and reproduce, or contest, certain relations of power and control from within the broader social domain.

As a result, communication in mathematics classrooms is part of the history and future trajectories of the pedagogic practice in all dimensions of content, pedagogy and assessment. Both Wenda Bauchspies and Dalene Swanson, in this volume, explain how communication in mathematics classrooms is influenced by discourses of the wider social context, and in what ways such influences are realized by participants (teachers and pupils) in the classroom practice. Their accounts urge us to challenge predominant assumptions about communication in mathematics classrooms. A deeper understanding of the politics of context is required, including: how dominant views about mathematical knowledge and its recontextualization in tasks and activities for use in the classroom are promoted and encouraged; how certain identities of mathematics learners are constructed or resisted; what the visions for the mathematics classroom pedagogic practice are; whose interests such a practice serves; and what the means for realizing such a vision are. All these dimensions play a significant role in how the nature of classroom communication evolves and develops.

In this volume, a number of chapters explore the changing (or not) of the mathematical activity in classroom practices based on attempts on the level of curriculum to change the practice of communication itself. For example, Tony Cotton argues tJiat adopting a *critical communication* pedagogic model can change drastically the classroom discourse to serve the interests of under-represented groups in terms of gender, race, and social class. In Chapter 10, he outlines a pedagogic model of critical communication, based on ideas offered in the perspectives of Bernstein, critical mathematics education, and social justice that can be used in mathematics classrooms. lie claims that the model can serve a vision of pedagogy in mathematics education that takes into account issues of gender, race and class and views mathematics teaching as the basis for personal and social development. Tony Cotton defines critical communication as the competency

> ...to critique reports that base their arguments on the use of mathematical data in the media. Similarly they can draw on mathematical arguments to counter claims from those that hold power over them. They can negotiate the processes through which they will come to increase their mathematical understandings, and they can analyze the administrative structures that construct a common sense view of what it is to learn mathematics.

The model includes aspects such as: seeing learning as a shared endeavor; exhibiting openness and honesty; planning for democratic, collaborative and inclusive decision making within the classroom; viewing pupil control of their learning as positive; showing trust of learners; giving care and responsibility to learners; expecting pupils to learn through challenge and variety.

Cotton makes a concerned effort to make his research democratic and useful to the students by trying to ensure "that every action I took to move my research forward should in itself be a worthwhile activity for the children involved." However, it remains a problem for research which wants to embrace issues around identity formation, subject positioning, class, race, gender, and so forth, that the researcher often writes from a position very different from those about whom he or she writes. We wish to see more research within this school of thought that engages with the methodological issues connected with voicing the silences in education and addressing issues of positioning in a way which embraces students' subjective realities.

FOCUS AND READING

This book is not meant to be an historic account or a complete review of related studies in die field. Neither is it a book that strives to promote a specific model for classroom communication. It is not a book about "effective communication" or "how to do things." Furthermore, it does not bring together a series of chapters reporting research on classroom communication per se. Instead, it offers a collection of studies (the contributed chapters in Part II) that analyze and discuss communication in mathematics classrooms from varied perspectives that approach communication within the complexity of the pedagogic context. It also provides reflections on the value and importance of these perspectives for understanding mathematics classroom communication (the reflective commentaries in Pari III).

We have grouped the chapters around the three themes through which we have discussed die significance of the contributions in the field of mathematics education and reviewed research studies in the area. This was to align them with what we consider an important strand of development in the field. Obviously, we could have chosen to group the contributions in other and radically different ways.

For example, an alternative grouping may be based on the theoretical roots and assumptions of each study. In this respect, some chapters reflect studies related to the trends of constructivism and socio-cultural theory, such as the chapter by Nadia Douek. Other chapters are studies that borrow concepts from the broader field of sociology. Specifically, Dalene Swanson addresses the sociology of mathematics education through the theories of Bernstein, Dowling and Walkerdine. Wenda Bauchspies draws on social anthropology, Tamsin Meaney on social semiotics and Tony Cotton on social jusuce. The rest of the chapters favour a more eclecdc use of theories in their analyses. Marcelo Borba uses ideas from Lev)' and

Tikhomirov to discuss the use of technology in communication in mathematics classrooms, and Dave Hewitt provides observations analyzed through the lens of an experienced teacher, seeing learning as related to the emotional, cognitive and social nature of the opportunities offered in the classroom interactions. Grouping the chapters this way provides us with a useful overview of the various theoretical frameworks that authors use to analyze and interpret their data, and it can provide new directions within each theoretical framework. However, it to some extent disguises possible interrelations and complementarities across theories. In contrast, our grouping within the three themes provides a better view of what contributions different theories have to offer on particular issues.

The question remains: how can the collection of research studies in this volume help us to see communication differently? What are the new answers or the new questions regarding the issues of communicating in, with and about mathematics? How do the contributions challenge the underlying assumptions and perceptions about communication in mathematics classrooms? What unanswered questions remain?

It is difficult to recognize shifts of paradigms in a field which is evolving and expanding in so many ways, and where so many theories exist and interact. What at first appear to be minor shifts in perspective, can later be recognized as major developments in the field. For instance, Anna Sfard (1996) has argued that while learning in older mathematics education texts is mainly seen as *acquisition* (in a particular non-participatory sense), newer texts reflect a *participation* view on learning. The participation perspective, seeing learning as increased participation in relevant practices rather than as acquisition of knowledge, has evolved gradually. This evolution is also reflected in the three themes discussed above. The chapters in this volume do not claim to represent any grand new theories or paradigm shifts, but they do reflect developments in the field over the past decades and as such produce a partial snap-shot of a field in movement.

Reading through the contributions in this volume, we witness an emphasis towards seeing communication in mathematics classrooms as part of pedagogic practice, as closely related to human interactions, and as part of striving to establish a particular discursive culture (e.g., promoting certain values such as collaboration, negotiation of mathematical meanings, dialogue, critical thinking, etc.). There is a noticeable shift from approaching communication as a toolkit of techniques and skills. Communication involves learners and teachers as whole human beings, encompassing not only talking and symbolizing but also feeling, valuing and imagining. The contributors argue that communication is closely connected with the contexts, the cultures and the practices of the mathematics classrooms. Their work challenges the underlying assumption of most mathematics classrooms that is based on a view of communication as unprobletnatic *transfer* of information (see Evans, 1999; Sfard, 1996). Further, they challenge predominant perspectives that approach mathematics teaching and learning through a "representational frame of mind," and lake a direction where the social and the individual are seen as connected. Finally, they reject assumptions that communication is either value-free or that interlocutors share similar values, ethics, and purposes.

In all, the contributions in this volume can be seen as raising a voice concerning research in mathematics classroom communication that includes at least the following dimensions:

a) *The role of media and innovative representational modes in changing palterns of communication in the mathematics classroom is not value-fee:* Marcelo Borba argues that media such as the graphic calculator always need to be seen in relation to the humans who use them. He sees their role as dynamic in changing classroom discourse (even when not using the technology). This contribution makes it clear how much we still have to learn about the influence of media on our discourses and practices. In contrast, Triadafilidis and Potaii notice that although the use of innovative representational media in the classroom encourage pupils to express themselves, the classroom communication patterns do not change. It seems that the role of the teacher (and the instructional interventions) remains an essential parameter.

b) *Communication needs deliberative organizing when the purpose is to develop mathematical competences or to challenge predominant views on mathematical knowledge.* In particular, Nadia Douek argues that productive communication as a foundation for conceptual development must be carefully organized, with the teacher taking a deliberative role. Inger Wistedt and Gudrun Brattstrom explain that a positive tone of communication in peer group settings is not enough to make university students embrace alternative perspectives concerning mathematical competencies. Interaction with a more knowledgeable other is seen as an essential parameter in the process of communication. It is certainly important to explore further those practices that promote productive communication and seriously take into account the development of mathematical knowledge.

c) *Culture, power, practice, identity and politics influence communication in tire matlrematics classrooms:* Tamsin Meaney presents a model to illustrate how any practice, including the work of mathematicians or pupils in a mathematics classroom, filters through the language/register used, with its cultural connotations necessarily forming the discourse. Wenda Bauchspies explains that power relations are reflected in the communication in mathematics classrooms, but at the same time the structures intended to maintain the status quo are cleverly manipulated by the students so as to gain some control over their positioning. Interestingly, this shapes these students' identities in new ways, perhaps counter to what they desired, namely as the more difficult learners who have managed to manipulate themselves into more desirable positions yet. do not seem capable of exploiting this opportunity in furthering their education. Dalene Swanson documents the extent to which the students are positioned and their identity forced into place through the communication with and about them. The remnants of Apartheid's "Fundamental Pedagogics" continues lo acquire bla< k students into its category of less able, though it is now disguised through being seen as a result of limited exposure to what are deemed relevant contextual experiences as well as language difficulties. Therefore, it appears a challenge to explore more relevant methodologies thai enable us to research how communication at the macro-social level of-sclvool and community influence communication at the micro level of classroom.

d) Mathematics classroom communication needs to be understood through ***the*** *participants' perspective:* This is embraced by Dave Hewitt, who acknowledges the interplay between students' emotional, cognitive and social reasons for how they choose to engage in mathematical activities. In a few cases, this has been linked to the inner somatic experience of engaging with mathematics (Slammert, 1993), or to the grounding of mathematical ideas in bodily experiences (Ntines, 2000). Tony Cotton's interest in doing research *for* the students is reflected in methodological choices where the students' perspective informs the research format. Whereas most work in the humanities wrestles with issues of objectivity and validity, as also touched on by Bauchspies, Cotton insists diat the methodology must reflect the social change aim of the research. This is an area which deserves more attention in mathematics education research. So far, Renuka Vithal (2000) provides a reflexive account of her engagement with a research methodology that respects the participants' perspective as well as reflects a social-cultural-political change perspective.

A NOTE ABOUT THE PROCESS

The idea for this monograph came about during one of Anna's visits to Aalborg University. There, she presented her perspective on constructivism to the doctoral students at the *Centre for Educational Development in University Science.* Anna's point was to look at constructivism as a discursive practice in education, rather than as a learning theory perse. Various theories or perspectives on education have, at various points in time, come to dominate educational discourse. While such theories/perspectives would not gain such prominent positions if they did not have a lot to offer, they at the same time frame and limit both the research and educadonal practices (through dialogue with die research as well as through managerial initiatives).

Through our discussions after Anna's presentation to the students, we came to acknowledge how research with a different starting point or applying a different perspective than what is considered mainstream, had been important to us. We could only agree that our task as researchers is not as much to provide guidelines for practice as it is to challenge the habitual. That implies bringing forward "alternative" discourses such as less prevalent perspectives. We have each worked on doing so in a number of areas. Madiematics classroom communication is one we bodi have worked within. It was Anna's suggestion to comprise a monograph with the aim of bringing forward alternative perspectives on mathematics classroom communication.

In the invitation for contributions we sent out in March 1999, we made the point that a prevailing research position in the field is to focus on a socio-cognitive approach where classrooms are analyzed as independent micro-cultures, communication is considered a rational process, mathematical content is not challenged, and students' intentions and values are not given due focus. We asked for papers which would direcdy challenge existing perspectives, theories or practices, or which offer novel perspectives. In particular, we requested papers with a socio-political approach to classroom communication, with an embodied approach to classroom communication or with novel methodological approaches. It was also our wish to have papers from countries less represented in the research literature. In other words, it was an ambitious project into which we ventured.

It has been a challenge to find less known contributors from Africa, South America, Asia and Eastern Europe. As Iben resides in South Africa, she had a number of potential authors in mind from the region, but they all declined the offer on grounds of being too engaged with teaching, administration, political work or completing thesis work. Even invitations to write joint papers were declined. Perhaps this reflects a general concern—potential contributors from countries in political and economical transition have too much on their plate to contribute to the international research community, which then continues to be dominated by white, male, European and North American voices. The community should be aware of this, as it means a loss to all. Our contributions do cover the areas we requested, though it becomes apparent that the socio-political approach still forms the most dominant alternative discourse. Thus we can see that our attempt to engage the community on challenging perspectives has only been partially fruitful.

The papers have been through a very extensive review process. Each paper was sent to two other authors, as well as to two or three external reviewers. In most cases, the reviewers managed to complete the task, so that each paper was reviewed by three to five people, besides ourselves and Leone Burton. The many rewrites took time, and often required more involvement from our side than we had originally expected. In the process, which now has lasted *3* years, we have had to undertake the immense extent of the editing process, as well as the difficulties that accompany bringing togedier our own very different positions in writing this introduction.

We also had to go through the most challenging, rewarding, time consuming and demanding times in our lives. Anna was the first of us to enter parenthood, when little Angelos joined her and Vasilis in January 2001. This was about the time when Iben fell pregnant with twins. Dylan and Keaton were born into her and Lionel's care in September 2001. Which was about the time that Anna became pregnant again. The final addition (so far) was Maria Rafaela, in July 2002. So while tlie production time of the monograph has been longer than anybody involved ever imagined (and certainly more than we would all have liked), it has indeed been a dine of production and contributing to the world in the most wonderful ways.

It has been a rewarding but time consuming task to put together this introduction. We have gone back and forth on the ways in which to organize the many approaches and perspectives prevalent in the field. Accordingly, we more than anybody know that the themes are our constructs, based on our reading of the research accumulated during the past decades, and our personal experiences and positions. Because of this, we have asked four prominent yet critical researchers in the field to comment *on* the chapters, and on the book as a whole. Stephen Lennan, Anna Sfard, Ilelle Alr0, and Ole Skovsinose have offered their commentaries in the final Part of the book.

We want to take this opportunity to thank first and foremost Leone Burton for offering us the editorial responsibility, and especially for her attempts to be patient with us and the frequent delays. Of course, we are most grateful to all the contributors, who have always been receptive to our comments and suggestions. Finally, we want to thank all the reviewers: Barbara Allen, Charlotte Krog Andersen, Bill Barton, Liz Bills, Morten Blomhoj, Laurinda Brown, Tony Brown, Guida tie Abreau, Simon Goodchild, Patricio Herbst, Yusuf Johnson, Henri Laurie, Roinulo Litis, Candia Morgan, Stig Andur Pedersen, Sal Restivo, Anna Sfard, Anna Tsatsaroni, Renuka Vithai, Rudolf vom Ilofe, and Vicky Zack. Without them, this monograph certainly would be less complete in more ways than one.

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